Numerical Averaging in Orbit Prediction

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This paper is a presentation of some useful aspects of a numerical averaging technique that has been applied with considerable success to the orbit prediction problem for artificial satellites. The method has been tested in a variety of situations wherein the satellite is subjected to combinations of perturbations for which adequate analytic theories are practically impossible. The technique provides many of the advantages of analytic averaging while retaining the ability to simulate the effects of any kind of small force that can be deterministically modeled. The theoretical basis for the method is straightforward but the details of its implementation are somewhat confusing and need some explanation. A particularly useful set of averaged equations is presented along with a description of an efficient algorithm for their solution. Examples of the advantages to be obtained by use of the method are included.

Nomenclature

a	= semimajor axis
e	= eccentricity
p	= semilatus rectum = $a(1-e^2)$
f	= true anomaly
i	= inclination
ω	= argument of pericenter
Ω	= longitude of ascending node
M	= mean anomaly
h	= $e \sin \omega$
k	= $e \cos \omega$
u	= $\omega + f$
Ξ	= $\omega + M$
r	= radius = $p/(1+e\cos f)$
R', S', and W'	= radial, circumferential, and normal components of disturbing acceleration
n	= mean motion = $(\mu/a^3)^{1/2}$
μ	= $k^2 m$ = universal gravitational constant times mass of central planet
τ	= orbital period = $2\pi/n$
df	= differential true anomaly = $r^{-2}(\mu p)^{1/2}\,dt$
$Q(e)$	= $[(1-e^2)^{1/2}-1]/e = -e/[(1-e^2)^{1/2}+1]$

Subscripts

1, 2	= value of the quantity at the lower and upper end of the averaging interval, respectively, $f_1 = f(t-\tau/2)$, $f_2 = f(t+\tau/2)$
0	= initial time

Introduction

FOR the past several years, numerical averaging has found increasing popularity and usefulness in orbit prediction work. Several investigators, when confronted with a problem for which no satisfactory analytic solution is known, have discovered that it is not necessary to resort to the very expensive special perturbation techniques used for high-accuracy work. Often, the averaging of the short-periodic terms of the perturbations can be accomplished by mechanical quadrature and, if the planetary equations are written in terms of disturbing accelerations, all kinds and combinations of small forces can be included. The incorporation of shadowing in long-term solar radiation pressure effects is no more difficult than the inclusion of an extra spherical harmonic coefficient. Low-thrust solar or nuclear electric propulsion is as easily included as third-body perturbations. The only assumption required is that the orbital elements do not change drastically during one revolution of the satellite in its orbit. Until the potential of computerized algebraic manipulations is more fully exploited, the development of analytic theories tailor-made for specific applications will remain a formidable task that can only rarely provide sufficient detail for realistic orbit mission planning. This is not to belittle that potential power but, rather, is intended to indicate a need for generally applicable approximate techniques for orbit prediction whose capabilities are not dependent upon the intuition and foresight of the developing theorist nor their usefulness upon the time required to develop an adequate theory by the usual analytic methods. The following sections present equations for the time rates of a particularly useful set of orbital elements, a description of the application of the averaging technique to those equations, some practical points on the implementation of the method, and some examples of its efficiency in the real-world orbit prediction problem.

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Planetary Equations in Gauss's Form

When the differential equations for the time rates of various sets of orbital elements are written in terms of the disturbing accelerations (rather than in terms of derivatives of a disturbing function), the equations are said to be in Gauss's form. The set selected for use with the averaging technique is given by

$$\dot{p} = 2(p/\mu)^{1/2} r S'$$
$$\dot{h} = (p/\mu)^{1/2}\{-\cos u R' + [(1+r/p)\sin u + (r/p)h] \times S' - k(r/p)\sin u \cot i W'\}$$
$$\dot{k} = (p/\mu)^{1/2}\{\sin u R' + [(1+r/p)\cos u + (r/p)k] \times S' + (r/p)h \sin u \cot i W'\}$$
$$di/dt = r\cos u W'/(\mu p)^{1/2} \tag{1}$$
$$\dot{\Omega} = r \sin u W'/[(\mu p)^{1/2} \sin i]$$
$$\dot{\Xi} = n - (p/\mu)^{1/2}\{[Q(e)][-\cos f R' + \sin f(1+r/p)S'] + (r/p)\sin u \cot i W' + 2(1-e^2)^{1/2}(r/p)R'\}$$

where the symbols have the meanings given in the notation section and the dot indicates the time derivative. The relationships given in Eq. (1) are completely equivalent to the actual equations of motion of the point mass spacecraft and can be integrated numerically or, in some cases, analytically to yield the time history of the position and velocity. Because of the appearance, however, of the fast variable f on the right hand of

Eq. (1), their solution, in the general case, involves very expensive numerical integration with a small computing interval.

The short-periodic (per-orbit-period) variations in Eq. (1) are the culprits that drive the high derivatives in rapid oscillation. Most analytic theories remove or separate the short-periodic variations by some means and obtain secular or long-period solutions that are analytically or numerically tractable. But these same theories nearly always make some kind of simplifying assumption concerning the nature of the disturbing function or the smallness of one of the parameters. In the next section it will be shown how these short-periodic variations can be removed from the differential equations by numerical quadrature and how the numerical averaging eliminates the need for additional analytic simplifications or assumptions.

The Averaging Technique

Let the time rate of change of any orbital element be represented by $\dot{E}$. The average time rate of the quantity E, over one revolution of the spacecraft in its orbit, is defined as

$$\bar{\dot{E}} = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} \dot{E}(t)\,dt \tag{2}$$

where τ is the orbital period defined in terms of the average value of the semimajor axis at time t as

$$\tau = 2\pi(\bar{a}^3/\mu)^{1/2}$$

Writing dt in terms of the true anomaly, f, in the Keplerian orbit transforms Eq. (2) into

$$\bar{\dot{E}} = np^2/[2\pi(\mu p)^{1/2}]\int_{f(t-\tau/2)}^{f(t+\tau/2)} (1+e\cos f)^{-2}\dot{E}(f)\,df \tag{3}$$

The Keplerian orbital elements are treated as constants during the averaging integration and can therefore be brought outside the integral sign. That is to say that the motion is assumed to be unperturbed during the averaging integration and the true anomaly is assumed to vary in the way described by Kepler's equation. Applying Eq. (3) to Eq. (2), we get

$$\bar{\dot{p}} = np^3(\mu\pi)^{-1}\int_{f_1}^{f_2}(1+e\cos f)^{-3}S'(f)\,df$$

$$\bar{\dot{h}} = np^2(2\pi\mu)^{-1}\int_{f_1}^{f_2}\{-\cos uR'(f)(1+e\cos f)^{-2} - k\sin u\cot i\,W'(f)(1+e\cos f)^{-3} + [(2+e\cos f)\sin u+h](1+e\cos f)^{-3}S'(f)\}\,df$$

$$\bar{\dot{k}} = np^2(2\pi\mu)^{-1}\int_{f_1}^{f_2}\{\sin uR'(f)(1+e\cos f)^{-2} + h\sin u\cot i\,W'(f)(1+e\cos f)^{-3} + [(2+e\cos f)\cos u+k](1+e\cos f)^{-3}S'(f)\}\,df \tag{4}$$

$$\overline{di}/dt = np^2(2\pi\mu)^{-1}\int_{f_1}^{f_2}\cos uW'(f)(1+e\cos f)^{-3}\,df$$

$$\bar{\dot{\Omega}} = np^2(2\pi\mu)^{-1}\int_{f_1}^{f_2}[\sin uW'(f)]/[\sin i(1+e\cos f)^3]\,df$$

$$\bar{\dot{\Xi}} = n - np^2(2\pi\mu)^{-1}\int_{f_1}^{f_2}\{\cot i\sin uW'(f)(1+e\cos f)^{-3} + 2(1-e^2)^{1/2}R'(f)(1+e\cos f)^{-3} - Q(e)[\cos fR'(f) - \sin f(1+r/p)S'(f)](1+e\cos f)^{-2}\}\,df$$

or

$$\bar{\dot{u}} = n - np^2(2\pi\mu)^{-1}\cot i\int_{f_1}^{f_2}\sin uW'(f)(1+e\cos f)^{-3}\,df$$

where it is to be understood that the integrals are now reduced to quadrature and that their values are to be determined by good mechanical quadrature formulae such as those named for Gauss or Lobatto. The disturbing accelerations (R', S', W') are calculated exactly as they would be in the osculating formulation. Indeed, the same subroutines can be used in each formulation; the distinction is that the spacecraft positions and the times of the derivative evaluations are defined by Kepler's equation and by the quadrature formulae in the averaging method. This point is the principal distinction between analytic and numerical averaging. It is this concept that eliminates the need for the additional approximation of the disturbing function that so often adds to the complexity and diminishes the accuracy of analytic techniques.

Equations (4), when used with good quadrature formulae and a stable numerical method for solving ordinary differential equations, provide a valuable tool for long-term orbit prediction and relieve the analyst of many a priori judgments that would be necessary with analytic averaging or expansion techniques.

Using the Method

The description above is sufficient to show the basic idea behind the method, but, as in the use of most approximate techniques, there are varying means of implementation corresponding to varying degrees of required accuracy. The following paragraphs are intended to display some of the practical experience gained by the author in several years of varied applications of the averaging technique. For many applications, the degree of detail presented below may not be required and it is recommended that the potential user carefully review the following recommendations and relate those suggestions to his particular needs.

Mean Elements

It is important, in the first place, to distinguish between the osculating orbital elements and the mean or averaged values of the elements. The Eqs. (4) describe the average time rate of change of the orbital elements but the right-hand sides have been written in terms of the osculating elements to show that we have a certain amount of freedom in specifying the mean values of the elements. In many cases where the method works well, it is sufficient to ignore the differences between the initial values of the osculating and mean orbital elements. To be more consistent, the system (4) should be written in terms of the mean values. That is to say that all the elements on the right, except f, should have bars over them to indicate that we are dealing with the mean rates of mean elements.

Further, we should notice that

$$(\overline{dE}/dt)_0 = 1/\tau\int_{t_0-\tau/2}^{t_0+\tau/2}\dot{E}(t)\,dt = [E(t_0+\tau/2) - E(t_0-\tau/2)]/\tau$$

and that, in terms of similarly defined mean values of the elements,

$$d\bar{E}/dt = d/dt\left\{1/\tau\int_{t-\tau/2}^{t+\tau/2}E(t)\,dt\right\} = \overline{dE}/dt \tag{5}$$

provided we maintain the understanding that the orbital elements remain constant during the averaging integration. Thus, by these few assumptions we have traded the system (1) of ordinary differential equations in terms of the osculating elements for the system (4) of ordinary differential equations in terms of the mean orbital elements.

Starting the Solution

The discussion above indicates that we should be concerned with obtaining a set of initial mean elements for use in the numerical integration of Eqs. (4). We require a consistent set of initial mean elements $\bar{E}_{i0}$ for use in starting the integration of Eqs. (4) which, under the assumptions represented by Eq. (5), now read

$$d\bar{E}_i/dt = f_i(\bar{E}_j, t), \qquad (j=1,6)(i=1,6)$$

The simplest way to obtain mean starting elements is to ignore this discussion and to use the given osculating values. The effect of such an assumption is a slow departure in phase and amplitude of the approximate solution from the mean of the actual solution. In some cases the given initial conditions may already represent a mean state as in the situation where data from a long-arc orbit determination are used to start the orbit prediction for subsequent analyses. In such situations, the direct use of the given initial conditions as starting values is probably better than the techniques outlined below.

A second method of starting the solution that is useful for close orbit problems near an oblate planet is to take account of the change in the mean motion due to the second zonal harmonic. The *mean* mean motion of the spacecraft in the main problem of artificial satellite theory is given[1] as

$$\bar{n}_0 = n_0\{1+(3J_2/2)(a_e^2/a_0^2)[(-\tfrac{1}{2}+\tfrac{3}{2}\cos^2 i_0)/(1-e_0^2)^{3/2}]+\cdots\}$$

to first order in the second zonal coefficient J_2. A corresponding value of the mean semimajor axis can be obtained by

$$\bar{a}_0 = (\mu/\bar{n}^2)^{1/3}$$

and the resulting orbit prediction will benefit from the incorporation of the major effect of J_2 on the mean motion. Further refinement is to be had by our knowledge of the long- and short-periodic effects of J_2 on the other elements but these are not so important as the variations in the semimajor axis.

For the sake of consistency, it is recommended that the major corrections for the effects of J_2 be applied to all the osculating elements if oblateness is a major factor. Some kind of procedure for removing the short-periodic variations in semimajor axis is a must for highly elliptic, close satellites of Earth, Mars, and Jupiter. The third, and most complex, method for obtaining initial mean elements is to perform a one-revolution numerical integration of Eq. (1) and, at the same time, form the integrals

$$\bar{E}_{i_0} = 1/\tau \int_{t=0}^{t=\tau} E_i \, dt \tag{6}$$

Now, according to the limits on the integrals of Eqs. (4), we must take the initial time to be $\tau/2$ and start the integration of Eqs. (4) with the quantities defined in Eqs. (6). This is the recommended approach for long simulations and should provide an excellent simulation of the long-term mean evolution of the orbit.

The careful reader may have noticed that the orbital period, τ, has been tacitly assumed known throughout the discussion above. It is a valid point to inquire whether the use of the initial osculating orbit period is the proper averaging interval for the starting procedure recommended above. If the short-periodic variations in semimajor axis are large, the following procedure should be used in the numerical startup. A running estimate of the average semimajor axis should be maintained and the starting integration represented by Eq. (6) should be terminated when the current time is equal to the period given by the (running) average semimajor axis. That is, at the end of each step in the numerical integration of Eq. (1), evaluate the quantity

$$\bar{\tau}(t) = 2\pi[\bar{a}^3(t)/\mu]^{1/2}$$

where $\bar{a}$ is determined from the running averaged values of $\bar{p}(t)$, $\bar{h}(t)$, $\bar{k}(t)$ given by

$$\bar{p}(t) = 1/t \int_0^t p(t)\, dt$$

and similarly for $\bar{h}$ and $\bar{k}$ with

$$\bar{a}(t) = \bar{p}(t)/(1-\bar{h}^2(t)-\bar{k}^2(t))$$

It should again be emphasized that these refinements are not essential to the use of the method and the complexities of the recommended numerical startup should be incorporated only after experimentation that shows a need for such sophistication.

The Averaging Integration

The evaluation of the definite integrals in Eqs. (4) can be performed in many ways and some numerical experiments in this area are quite valuable in the evaluation of the requirements for specific applications. The author has made satisfactory use of gaussian quadrature integration in several applications of varying complexity. The integration formula named for Lobatto may be useful in situations where it is particularly important to obtain the values of the integrals in Eqs. (4) at the endpoints of the averaging interval.

The averaging interval can be broken up into several subintervals if more information about the behavior of the integrand is required. The number of intervals and the order of the quadrature formula for each interval are variables that should be adjusted to suit the particular application. The author has found that three intervals of 6th order gaussian quadrature are sufficient for nearly any application. In many cases, two intervals with 6 ordinates each provide excellent results while, in some cases, a single 6 point quadrature is adequate.

The potential user should maintain control over the number of ordinates and intervals and should experiment carefully to get the full benefit of the averaging method. The weights and abscissas for quadrature formulas up through order 9 or 10 should be stored and the number of segments into which the averaging interval can be broken should be made variable so that these parameters can be nearly optimized for various applications. In coding the method, the programer should make it easy to redefine the interval of averaging. There seems to be no reason why the technique cannot be used for obtaining the long-term effects of resonance phenomena by simply averaging over several revolutions, depending on the nature of the resonance, instead of just one revolution. The author hopes to be able to report on this kind of application in the near future.

Integration of Averaged Equations

The removal of the high-frequency variations from the planetary equations permits the use of a large computing interval in the numerical integration of Eqs. (4). Each analyst has his own favorite method for solving systems of ordinary differential equations and most analysts will use that favorite method in spite of what may be suggested here. It is, however, suggested that the potential user perform experiments with several integrators before deciding on any one technique. The results of such studies will depend markedly upon the nature of the averaged disturbing function and it is the author's opinion that there is no single "best" method. It is recommended that studies be conducted with single and multistep methods of order 6 to 10 for each basically different orbit prediction problem of interest.

Now that we have evaluated the derivatives 6 to 20 times during the averaging integration, we must be able to take fairly sizable steps to show a significant advantage over straight numerical integration of Eqs. (1). The size of the maximum computing step that can be used in solving Eqs. (4) depends upon the second highest frequency in the disturbing function (the highest frequency in the averaged disturbing function). It is this frequency that controls the ultimate advantage of the method as presented here. Thus, for a lunar orbiter whose period is about three hr, the required step-size will be controlled by the 14-day terms in the lunar gravity disturbing function and we can use a step of from one to four days in the integration of Eqs. (4). In practice, the averaging gives a 10- to 15-fold speed advantage if the averaging integration uses 18 points per orbit for the L1 lunar field model. It is important to note that if the moon were spinning at a slower rate, this second-highest frequency would be lower and a larger step could be taken. For other applications where the tesseral gravity components are unimportant, the step can be increased to yield as much as a 100-fold speed advantage over the full numerical integration of (1). The advantage realized for any given application will depend upon the ability of the analyst to exploit the absence of the short-periodic variations and upon the complexity of the averaged disturbing function. The remarks above are meant to be helpful hints rather than rigid statements of definition. A thorough knowledge of the major perturbations and their relative frequencies and magni-

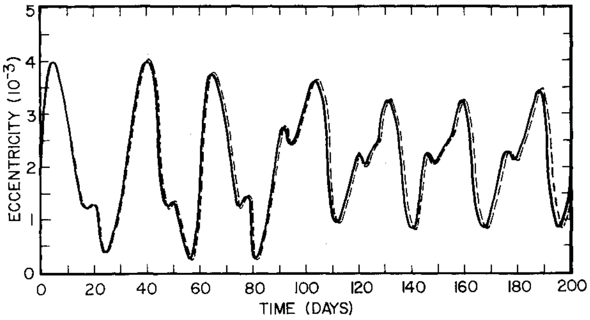

Fig. 1 RAE-B eccentricity history.

tudes is much more valuable than a set of rules to use in the implementation of the averaging technique.

Some Applications

In this section, a few applications of the numerical averaging technique are described. The examples are of sufficiently distinct situations that the presentation tends to show the versatility of the method. Each example is accompanied by a short discussion of the particular problem and its distinguishing features.

Lunar Satellite Orbit Evolution

This is the most exacting application with which the author has direct practical experience. The problem is to predict the long-term motion of the Radio Astronomy Explorer (RAE-B) whose objective is to study the electromagnetic celestial sphere using an interferometer consisting of 250 m boom antennas stabilized by the lunar gravity gradient.

Because of attitude stability requirements imposed by the long flexible V antennas, it is important that the orbit remain essentially circular throughout the useful lifetime of the spacecraft. The 1100 km altitude of the desired orbit is sufficiently low that the elusive lunar gravitational harmonics cause significant variations in the orbital elements. In particular, the second and third degree harmonics interact to cause a complicated motion in the eccentricity that is the subject of some concern to mission planners in the face of our inability to determine an adequate model of the lunar gravity.

Figure 1 shows the time history of the eccentricity for a lunar orbit like the one desired for the RAE-B mission. The initial conditions and other information necessary to reconstruct the computation are given in Table 1. The dynamic model includes the L1 model of the lunar gravity along with the Earth and the sun whose positions were obtained from the JPL Development Ephemeris 69(DE69). The figure shows the actual motion (solid curve) obtained by a precision numerical integration of the actual equations of motion and the motion given by integration of the averaged equations (4). The dashed curve gives the approximate solution in the situation where the integration of Eqs. (4) was started with osculating orbital elements as initial conditions.

The inclination of the orbit has the value at which the second zonal harmonic causes no first order secular motion in the argument of pericenter. It is for orbits having this "critical" inclination that many approximate theories break down. Murphy et al.[2] have considered the long-term motion of the eccentricity for RAE-B type orbits and have concluded that the inclination should be near the critical value if long-periodic growth of the eccentricity is to be suppressed. The numerical averaging, here, is similar to special perturbation techniques in that the applicability of the method is not dependent upon the specific value of the inclination.

It is difficult to make an accurate evaluation of the speed advantage of one method over another without very carefully qualifying each point of comparison. For the RAE-B lunar orbit problem, the advantage in speed of the averaging technique over full numerical integration ranges from 7- to 20-fold depending upon the level of accuracy desired in the comparisons. It would be possible, by comparing a high precision numerical integration to a crude averaging integration, to show a speed advantage of perhaps a hundred but the comparison would be misleading and, in the author's opinion, dishonest. The averaging is not a super-method but is, rather, a versatile technique that lies, in speed and accuracy, between the classical analytic methods and full scale numerical integration. A typical averaging integration for a 200 day RAE-B orbit prediction requires about 1 min on the IBM 360/91 while the same prediction with full integration requires about 8 min.

Geosynchronous Orbit Raising Mission

The second example is for a mission that has received some consideration during the past year. The problem is to predict the motion of a spacecraft during its ascent from close Earth orbit up to synchronous altitude under the influence of a power-limited propulsion system deriving its energy from the sun's light falling on solar cells. This geosynchronous orbit mission using solar electric propulsion might be called GEOSEP to save time in referring to the concept.

The basic energy relationships for such a mission can be analyzed by fairly straightforward analytic averaging as Edelbaum[3] has shown. When more detailed analyses are required, however, the analytic representation of the many perturbing effects becomes difficult and numerical averaging can provide a valuable intermediate step between analytic solutions describing only the low-thrust effects and full numerical integration describing the other perturbations as well.

Figure 2 shows a simple example of a GEOSEP ascent trajectory with the combined effects of constant solar electric circumferential thrust, Earth zonal harmonics J_2, J_3, and J_4, and the luni-solar perturbations. The solid curve shows the time history of the semimajor axis and the dashed curve displays the motion of the orbit's ecliptic node. The simulation of several disturbing forces is no more difficult than the simulation of one,

Table 1 RAE-B Lunar orbit initial conditions

Epoch	J.D. 2439305.7170	
$a_0 = 2838.0$ km	$\omega_0 = 0°$	w.r.t. true lunar
$e_0 = 0.00085°$	$i_0 = 116.5°$	equator and prime
$f_0 = 0°$	$\Omega_0 = -71.8318°$	meridian of epoch
Quadrature integration		
Intervals per revolution = 3		
Ordinates per interval = 6		
L1 model of lunar field (unnormalized)		
$C_{20} = -2.07108 \times 10^{-4}$	$C_{22} = 2.0716 \times 10^{-5}$	
$C_{30} = 2.1 \times 10^{-5}$	$C_{31} = 3.4 \times 10^{-5}$	
$C_{33} = 2.583 \times 10^{-6}$		

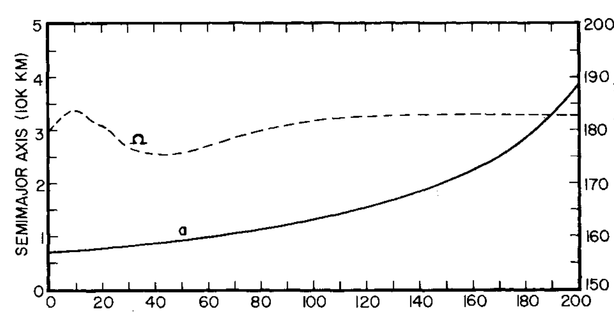

Fig. 2 GEOSEP orbit raising mission.

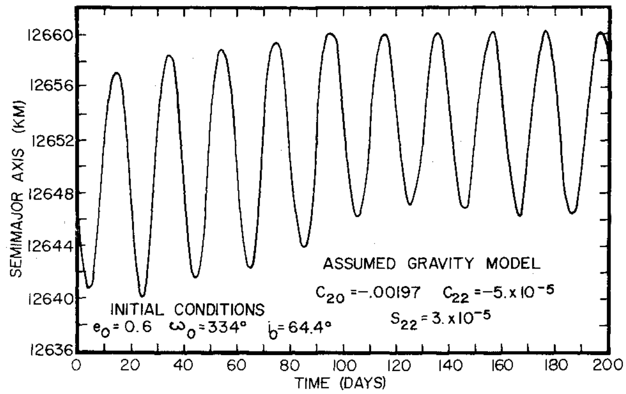

Fig. 3 Subsynchronous Mars orbiter.

a point that has important implications for the usefulness of the method. The simulated solar electric propulsion system is typical of those being considered for actual use. A 20 kw power plant driving an ion engine of specific impulse 2000 sec was assumed for a spacecraft with initial mass of 3000 kg.

Here is an excellent application for the averaging technique. Any number of small forces can be included in the simulation as well as the effects of power degradation due to radiation during ascent through the Van Allen belts and power loss during shadow passages. The important point is that these effects can be included without the need for extensive restructuring of earlier assumptions and/or computer software as would be required with the usual analytic methods.

Resonance—Subsynchronous Mars Orbiter

The third application of the averaging technique is to a resonance problem. Resonance, with its small divisors, is probably the most confusing problem of celestial mechanics. Except for the real, physical complexities of resonance motion, there are no difficulties associated with the use of the averaging technique for most resonance problems. This is not meant to imply that numerical averaging will work for all resonance problems but that, in many cases, a significant advantage can be obtained by its use. The method is especially valuable in situations where the periods of the resonance terms are large in comparison with the orbit period.

Figure 3 shows the time history of the mean semimajor axis for a Mars orbit similar to the one occupied by the Mariner 71 spacecraft. The orbit period is about 12 hr and the 2 to 1 resonance with the Martian equatorial ellipticity is characterized by periodic variations in the mean semimajor axis. These variations are accompanied by changes in the orbital period of the order of one minute. Energy is being transferred back and forth from spacecraft to planet and the tradeoff effect is most obvious in the periodic variations in semimajor axis.

In the example of Fig. 3, the frequency of the principal resonance argument is just low enough that a computing step of several revolutions can be used in the integration of the averaged equations. In this application, the author obtained a speed advantage factor of only three or four over full numerical integration. If the frequency of the principal resonance term were much higher, it would probably be more efficient to use full integration than to use the averaging technique. This is typical of the performance of the method in resonance applications and the potential user should always check to make sure that the lowest frequency in the averaged disturbing function is at least three or four times lower than the orbital frequency before attempting to apply the averaging method.

The suggestions above apply equally well to all applications of the method and not just to resonance. Normal caution in coding the method and ordinary consideration for basic energy relationships are sufficient to guide the analyst in nearly any application.

Conclusions

The foregoing sections have included a description of a numerical averaging technique for long-term orbit prediction, some explanatory remarks on the implementation of the method, and a few examples of its use in the practical world of orbit mission analysis.

Emphasis was placed on the role of the method as an intermediate analysis tool lying somewhere between completely analytic and the usual special perturbation techniques. It was pointed out that the method can include all kinds and combinations of small forces in the case where the small forces do not drastically change the orbital elements during one revolution of the satellite in its orbit. It was further pointed out that the method is not always superior in speed to full numerical integration and that the analyst should gain a knowledge of the principal frequencies in the averaged disturbing function before attempting to use the technique.

The section on applications tends to demonstrate the versatility of the method. Applications are shown that demonstrate the effectiveness of the averaging in close satellite or geodesy problems, low-thrust orbit-raising problems, and situations where resonance effects play an important part. It was suggested that, by careful selection of the averaging interval, the technique might be useful in situations where multiple or odd-order resonance effects are important.

The author has found the averaging technique so useful in such varied applications that it seemed important to write down a description of the basic formulation so that others who have not already discovered the method might benefit from the exposition. It is the author's wish that the descriptions and suggestions in this paper will be useful in the future exploration of space.

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